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Prepared for
Electronics Research Directorate
Air Force Cambridge Research Laboratories
Air Force Research Division
Air Research and Development Command
United States Air Force
Bedford, Massachusetts

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INTERPOLATION TECHNIQUES
APPLIED TO PATTERN SYNTHESIS

A. Ksienski

Scientific Report No. SR 3508/R

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TABLE OF CONTENTS

ABSTRACT	iii
I. INTRODUCTION	1
II. TRIGONOMETRIC INTERPOLATION	1
III. ERROR ANALYSIS	5
IV. EXAMPLES	12
V. CONCLUSIONS	16
APPENDIX I: Computation of the Coefficients of a Fourier Interpolation Sum of 6 Terms Approximating a Sector Beam (See Figure 2)	16
APPENDIX II: Approximate Characteristics of Woodward Synthesis for Continuous Apertures	17
REFERENCES	19

ABSTRACT

A pattern synthesis method is presented that has approximative characteristics similar to that of a Fourier series but is computationally much simpler, and consequently results in a higher over-all accuracy. Error estimates and convergence measures are presented, which compare the above synthesis to the Fourier approximation and the Woodward method.

I. INTRODUCTION

The two most important requirements in pattern synthesis are minimization of error and the ease of computation in obtaining the excitation coefficients. Sometimes one of the two has to be sacrificed for the other. Thus, of the two most prevalent methods, the Fourier expansion is the synthesis procedure that results in a known error which is also minimized in the mean square sense. The difficulty in obtaining the excitation coefficients is often quite serious, particularly if one has to include the element factor in the expansion. On the other hand, the Woodward synthesis procedure is quite easy computationally but lacks error estimates for points that fall between the selected beam positions for which the pattern is exactly matched. It is thus impossible to predict the required number of array elements, or the aperture width, to meet given error specifications, and in fact, the question arises whether the error decreases at all for a larger number of terms in the approximating sum.

Most of these problems can be solved if one uses trigonometric interpolation. The technique is both straightforward and computationally easy and accurate. The estimated maximum error is of the same order of magnitude as that of the Fourier approximation, but the actual error may sometimes exceed the Fourier error and sometimes be smaller.

II. TRIGONOMETRIC INTERPOLATION

The method of interpolation is rather well known, but it appears that it has been overlooked in many applications. One reason is that often poor results have been obtained from this method; another is the difficulty of inverting a large matrix and the inherent numerical inaccuracies incurred in such inversion even with the aid of a computer. The cause of both problems lies in the fact that the various terms of the polynomials usually used for interpolation, namely powers of x , are not mutually orthogonal. The large errors of approximation can be reduced by introducing a non-uniform spacing between interpolation points which if properly chosen will result in a Tchebyscheff polynomial. But the inversion difficulty still remains.

In the case of a trigonometric or exponential polynomial the various terms are mutually orthogonal, not only with respect to the integral over the period of orthogonality but also with respect to the sum. Thus if the functions ϕ_i and ϕ_k are mutually orthogonal we have

$$\sum_{a=1}^n \phi_i(x_a) \phi_k(x_a) = 0 \quad \text{for } i \neq k \quad (1)$$

where x_a are the interpolation points covering the orthogonality period as in the integration case. The expansion coefficient is given by

$$c_i = \frac{y \cdot \phi_i}{\phi_i \cdot \phi_i} = \frac{\sum_{a=1}^n y_a \phi_i(x_a)}{\sum_{a=1}^n \phi_i^2(x_a)} \quad (2)$$

where y_a is the value of the approximated function at x_a . That this is the correct value can be seen from the following. Let

$$\sum_{j=0}^N c_j \phi_j(x_a) = y_a \quad a = 1, 2 \dots n \quad (3)$$

which defines the interpolation values and their location. Multiplying both sides by $\phi_i(x_a)$ and summing over a , we obtain

$$\sum_{a=1}^n \phi_i(x_a) \sum_{j=0}^N c_j \phi_j(x_a) = \sum_{a=1}^n \phi_i(x_a) y_a \quad (4)$$

Interchanging the order of summation on the left, namely, summing over a first, we obtain

$$\sum_{j=0}^N \sum_{a=1}^n \phi_i(x_a) \phi_j(x_a) c_j = \left(\sum_{a=1}^n \phi_i^2(x_a) \right) c_i \quad (5)$$

using the orthogonality given in equation (1). Thus,

$$c_i \sum_{a=1}^n \phi_i^2(x_a) = \sum_{a=1}^n y_a \phi_i(x_a) \quad (2a)$$

which is equivalent to equation (2).

In the case of complex $\phi_i(x)$ the expression for c_i will be given by

$$c_i = \frac{\sum_{a=1}^n y_a \phi_i^*(x_a)}{\sum_{a=1}^n |\phi_i|^2(x_a)} \quad (2b)$$

where ϕ_i^* is the complex conjugate of ϕ_i . The orthogonality condition is given by

$$\sum_{a=1}^n \phi_i(x_a) \phi_k^*(x_a) = 0 \quad (1b)$$

The summation $\sum_{a=1}^n |\phi_i(x_a)|^2$ is the normalization constant for the vector $\phi_i(x)$.

For an array of $2N + 1$ isotropic elements spaced $\lambda/2$ apart, the set of functions forming our orthogonal function space are $\{e^{jn\pi u}\}$ where $u = \sin \theta$. The expression representing the array is given by

$$P(u) = \sum_{n=-N}^N a_n e^{jn\pi u} \quad (6)$$

and the excitation coefficient a_n is given by,¹

$$a_n = \frac{1}{2N} \sum_{a=-N}^N y_a e^{-jn\pi \frac{a}{N}} \quad (7)$$

where the prime \sum' implies that the first and last ordinates (y_{-N} , y_N) are taken with half their value. Also, the first and last excitation coefficients are applied with half of their computed value.¹ The reason for this weighting factor which is applied to the end forms may be seen from the expression

$$\frac{1}{2} e^{-jn\theta} + e^{-j(n-1)\theta} + \dots + e^{j(n-1)\theta} + \frac{1}{2} e^{jn\theta} = \sin n\theta \cot \theta/2 \quad (8)$$

which vanishes for $\theta = \frac{\pi m}{n}$, where $m = 0, 1, 2, \dots$. If $\phi_l = e^{j(l\pi x)}$ then

$$\phi_l(x_a) \cdot \phi_k^*(x_a) = e^{j(l-k)x_a} = e^{j(l-k)\frac{\pi a}{n}} \quad (1c)$$

and

$$\sum' \phi_l(x_a) \phi_k^*(x_a) = \sum_{a=-n}^n e^{j(l-k)\frac{\pi a}{n}} = 0 \text{ for } l \neq k \quad (9)$$

which satisfied the orthogonality property with the above weighting. One actually may omit the end element and construct the interpolation without it, namely, use $2N$ elements covering the interval of 2π ($\frac{2n}{2n+1}$) of the interpolation range, the reason being that the value of the ordinate at 2π is identical with its value at zero. The sum corresponding to equation (8) will be

$$1 + e^{j\theta} + e^{j2\theta} + \dots + e^{j(2n-1)\theta} = \frac{1 - e^{j2n\theta}}{1 - e^{j\theta}} = 0 \text{ for } \theta = \frac{m\pi}{n}. \quad (10)$$

Thus

$$\sum \phi_l(x_a) \phi_k^*(x_a) = \sum_{a=0}^{2n-1} e^{j(l-k)\frac{\pi a}{2n}} = 0 \text{ for } l \neq k \quad (11)$$

In the case that $y(u)$, the desired function, is even or odd, the corresponding expressions for the coefficients will be simplified and given by

$$a_n = \frac{2}{N} \sum_{a=0}^N y_a \cos\left(na \frac{\pi}{N}\right) \quad (12)$$

for an even $y(u)$, where

$$y(u) \simeq \frac{1}{2} a_0 + a_1 \cos \pi u + a_2 \cos 2\pi u + \dots + a_{N-1} \cos(N-1)\pi u + \frac{a_N}{2} \cos N\pi u \quad (13)$$

For an odd $y(u)$

$$b_n = \frac{2}{N} \sum_{a=1}^{N-1} y_a \sin na \frac{\pi}{N} \quad (14)$$

where

$$y(u) \simeq b_1 \sin \pi u + b_2 \sin 2\pi u + \dots + b_{N-1} \sin(N-1)\pi u \quad (15)$$

Note that the cosine expansion has the same characteristic as the exponential, in that the first and last terms appear with half value both in the expansion and in the computation of the coefficients of excitation (see Appendix I). Thus it is seen that the solution of the interpolation problem does not involve any matrix inversion, therefore ensuring that no loss of accuracy occurs throughout the computational process.

III. ERROR ANALYSIS

The problem of error estimates is becoming more important because of the stringent requirements of modern applications. And in view of the increasing size of arrays used the questions of convergence of the representing series are also becoming relevant. It was therefore considered of importance to discuss these characteristics as they apply to the three methods mentioned above, namely, the Fourier approximation, the trigonometric interpolation, and the Woodward method. Since the Woodward method is presently in wide use, a particular attempt was made to investigate the convergence properties of this method and also to estimate its approximation error as compared with the Fourier method.

The Fourier approximation and the trigonometric interpolation have the same convergence properties which can be expressed as follows:² If $f(x)$ is continuous with modulus of continuity $\omega(\delta)$ (which means that $\omega(\delta) = \max |f(x_2) - f(x_1)|$ for $|x_2 - x_1| \leq \delta$) and $s_n(x)$ is the Fourier n^{th} partial sum, then

$$|f(x) - s_n(x)| \leq A \omega\left(\frac{2\pi}{n}\right) \log n \quad (16)$$

If $\omega(\delta) \log \delta \rightarrow 0$ as $\delta \rightarrow 0$, then $s_n(x) \rightarrow f(x)$ uniformly. If $f(x)$ has a p^{th} derivative $f^{(p)}(x)$ such that

$$|f^{(p)}(x_2) - f^{(p)}(x_1)| \leq \lambda (x_2 - x_1) \quad (17)$$

for all x_1 and x_2 , λ being constant, then

$$|f(x) - s_n(x)| < \frac{A_p \lambda \log n}{n^{p+1}} \quad (18)$$

or

$$|f(x) - s_n(x)| \leq \frac{A_p}{n^p} \omega\left(\frac{2\pi}{n}\right) \log n \quad (19)$$

where A_p is an absolute constant depending on p alone. The same estimate holds for a trigonometric interpolating sum of the same number of terms. Thus the convergence properties of the two approximation methods are identical. This similarity holds as well for piecewise continuous function.

Now let us consider the Woodward approximation. For a discrete array of $2N + 1$ elements spaced $\lambda/2$ the Woodward expression would be

$$P(u) = \sum_{n=-N}^N A_n \frac{\sin \left[\frac{(2N+1)\pi}{2} \left(u - \frac{2n}{2N+1} \right) \right]}{\sin \frac{\pi}{2} \left(u - \frac{2n}{2N+1} \right)} \quad (20)$$

which when expanded is equal to

$$P(u) = \sum_{n=0}^N a_n \cos n\pi u + b_n \sin n\pi u \quad (21)$$

which is a trigonometric sum of the n^{th} order. This sum equals the desired function at exactly $2N + 1$ points spaced at distances of $\frac{2\pi}{2N+1}$ from each other. This, however, is by definition an equispaced trigonometric interpolation and as such should have again the same convergence properties as the Fourier expansion. Thus the convergence properties of all three methods are the same.

The next question is that of the error estimate. This problem is very difficult if a useful estimate is wanted, namely, within a few percent of the actual error. This is the reason why equations (16)-(19) cannot

serve this purpose and are good only as a convergence criterion when a large number of terms are involved. The difficulty lies in finding the lowest upper bound on the constant A or A_p in each of the cases mentioned above. A constant which is close to this optimum is available for the Fourier approximation of continuous functions and is given by³

$$A = \frac{2}{\pi^2} \quad (22)$$

so that the error estimate in the maximum absolute deviation sense is given by

$$E_n < \frac{2}{\pi^2} \frac{\log n}{n^p} M_p \quad (23)$$

where n is the order of the corresponding Fourier partial sum, and M_p is the maximum value of the p^{th} derivative of the approximated function. This estimate seems to hold even when applied to functions containing discontinuities except in the vicinity of the discontinuity where the occurrence of the Gibb's phenomenon will produce a deviation of approximately 9 percent of the total jump in the function. * This deviation is independent of n . This phenomenon can be eliminated by the modification of the Fourier coefficients in a manner which is equivalent to a Fejer summation.³

For the trigonometric interpolation process such minimal constant is not available, but it can be shown that the error in the interpolation method is very close to that of the Fourier approximation. It will be shown that the difference between the two is in fact that the Fourier partial sum of n^{th} order completely neglects higher order components, while in the interpolation method the higher order components get reflected and add on to the lower order components. However, for a good Fourier approximation the deviation of the n^{th} partial sum from the desired function is very small; this implies that the higher order components are negligible, and hence it makes little difference whether their amplitudes are recorded or not. Also, one should remember that although any modifications of the Fourier

*Reference 4, page 105.

components increase the mean square error,* they do not necessarily increase the maximum deviation; in fact, in some cases they decrease it. One such example is the tapering of higher order Fourier components which was shown to lead to a smaller maximum deviation than that of the equivalent Fourier partial sum.³ Let us consider in detail the effect of substituting the interpolation method for the Fourier approximation. If the approximated function $f(x)$ has only the first N Fourier components, both methods will yield the function exactly. In the case of higher order components they will be recorded in the following way: consider the $(N+k)^{\text{th}}$ Fourier component of the m^{th} term in the interpolation sum ϕ_m^{N+k} , and assume we have an odd function such that only sines are present.

$$\phi_m^{N+k} = \sin(N+k)x \sin mx \quad (24)$$

$$= (\sin Nx \cos kx + \cos Nx \sin kx) \sin mx$$

for

$$x = a \frac{\pi}{N} \quad a = 1, 2 \dots N-1$$

$$\phi_m^{N+k} = (\cos a\pi \sin k \frac{a\pi}{N}) \sin mx \quad (25)$$

Now consider the $(N-k)^{\text{th}}$ Fourier component of the m^{th} term

$$\phi_m^{N-k} = \sin(N-k)x \sin mx \quad (26)$$

*The mean square error is equal to

$$\frac{1}{2\pi} \int |f(x) - \sum a'_n e^{jnx}|^2 dx = \frac{1}{2\pi} \int |f(x)|^2 dx + \sum a_n'^2 - 2 \sum a_n a_n'$$

where a_n is the Fourier coefficient and $a_n' = a_n + \epsilon_n$, and $f(x)$ is the approximated function. Substituting for a_n' we get

$$\begin{aligned} \frac{1}{2\pi} \int |f(x)|^2 dx + \sum a_n' (a_n' - 2a_n) &= \frac{1}{2\pi} \int |f(x)|^2 dx + \sum (a_n + \epsilon_n)(\epsilon_n - a_n) \\ &= \frac{1}{2\pi} \int |f(x)|^2 dx - \sum a_n^2 + \sum \epsilon_n^2 \geq \frac{1}{2\pi} \int |f(x)|^2 dx - \sum a_n^2 \end{aligned}$$

where the equality sign holds only for $\epsilon_n = 0$.

which for $x = a \frac{\pi}{N}$ results in

$$\phi_m^{N-k} = -(\cos a\pi \sin k \frac{a\pi}{N}) \sin(mx) = -\phi_m^{N+k}$$

Thus it is seen that the higher odd harmonic components of the function $f(x)$ will be recorded with reversed sign as lower components. The even harmonics are recorded with the same sign as can be seen from the following equation,

$$\psi_m^{N+k} = \cos(N+k)x \cos mx = \cos a\pi \cos k \frac{a\pi}{N} = \cos(N-k)x \cos mx = \psi_m^{N-k}$$

for $x = \frac{a\pi}{N}$ (27)

As mentioned above, if the higher order components are negligible in magnitude, their effect will be just as negligible in the Fourier series as it is in the interpolation case. In the case that these components are significant the implication is that the Fourier approximation will have a significant error. In this case the interpolation will also suffer from a similar although not identical error, and the maximum deviation error will be somewhat smaller in some cases and somewhat larger in others, but it will be of the same order of magnitude as that of the Fourier approximation.

Let us now investigate the error involved in the Woodward technique. As mentioned, the Woodward technique is an interpolation method with its interpolation points slightly closer together than in the method previously discussed. Thus for an N^{th} order trigonometric sum the spacing is $\frac{2\pi}{2N+1}$ in the Woodward as compared to $\frac{\pi}{N}$ for the former one. One can intuitively say that the two methods should yield almost the same results when N is relatively large, and assuming that $f(x)$ is continuous. We shall try to show in a more rigorous fashion that the two interpolations are quite similar and also indicate in what way they differ.

To simplify the discussion we will assume that $f(x)$ is an even function of x ; this does not result in loss of generality since a similar proof may be used for an odd function and a sum of the two can represent any arbitrary function. Since $f(x)$ is even, it is represented by a summation of cosines alone. Thus, let the regular interpolation sum discussed previously be represented by $P_1(x)$ and the Woodward sum by $P_2(x)$; hence

$$P_1(x) = \sum_{n=0}^N a_n \cos nx \quad (28)$$

where $P_1(x) = f(x)$ at $x = a \frac{\pi}{N}$ $a = 0, \pm 1, \dots \pm N$

$$P_2(x) = \sum_{n=0}^N b_n \cos nx \quad (29)$$

where $P_2(x) = f(x)$ at $x = a \frac{2\pi}{2N+1}$ $a = 0, \pm 1, \dots \pm N$

Let us now substitute powers of $\cos x$ for $\cos nx$ such that

$$P_1(x) = \sum_0^N a_n \cos nx \equiv \sum_0^N A_n \cos^n x \quad (30)$$

and

$$P_2(x) = \sum_0^N b_n \cos nx \equiv \sum_0^N B_n \cos^n x \quad (31)$$

Now substituting y for $\cos x$ we obtain,

$$P_1(y) = \sum_0^N A_n y^n \quad (32)$$

and

$$P_2(y) = \sum_0^N B_n y^n \quad (33)$$

where

$$0 \leq x \leq \pi \quad +1 \geq y \geq -1$$

We now have two polynomials of y interpolating the function $f(y)$ where $y = \cos \pi x$. This amounts to a Lagrangian interpolation whose characteristics are well known and where the error, or remainder, is given by

$$R(y) = \frac{\omega(y)}{(N+1)!} f^{N+1}(\xi) \quad (34)$$

where

$$\omega(y) = (y - \rho_0)(y - \rho_1) \dots (y - \rho_n) \quad (35)$$

and where the ρ_n are the interpolation points. At these points the function is equal to the polynomial and therefore the error must vanish. The value ξ represents a point close to the middle of the interpolation interval of $P(y)$, and is connected with the location of ρ_n .

Now let us compare the two remainder expressions

$$R_1(y) = \frac{(y-1)(y - \cos \frac{\pi}{N})(y - \cos \frac{2\pi}{N}) \dots (y+1)}{(N+1)!} f^{N+1}(\xi_1) \quad (36)$$

and

$$R_2(y) = \frac{(y-1)(y - \cos \frac{2\pi}{2N+1}) \dots (y - \cos \frac{2\pi N}{2N+1})}{(N+1)!} f^{N+1}(\xi_2) \quad (37)$$

The difference between the expression is in ξ and in the ρ_n . For a fairly smooth function the variations of $f^{N+1}(x)$ are limited. The change in ξ is directly related to the distribution of ρ_n , and a small shift in those points will produce a correspondingly small change in ξ . Exaggerating this shift by assuming that all points moved by an amount:

$$\frac{\pi}{N} - \frac{2\pi}{2N+1} = \frac{\pi}{N(2N+1)} \quad (38)$$

we obtain that the change in $f^{N+1}(\xi)$ is given by

$$\left| f^{N+1}(\xi_1) - f^{N+1}(\xi_2) \right| \leq \left| f^{N+2}\left(\frac{\xi_1 + \xi_2}{2}\right) \frac{\pi}{N(2N+1)} \right| \quad (39)$$

The difference in the ρ_n 's is reflected in the values of $\omega(y)$. Now for most of the interval of approximation the zeros of $\omega_2(y)$ are more closely spaced than that of $\omega_1(y)$; hence, the magnitude of $\omega_2(y) \leq \omega_1(y)$ for this range. However the error will increase in the vicinity of $y = -1$ ($x = \pi$), since a zero located there was shifted to $x = \frac{2N\pi}{2N+1}$. The value of the error may be calculated if desired by computing the total effects of the zero shifts. From a practical point of view this often is of no importance since the element factor may produce a zero at precisely that location, which is the end of the visible range. Thus for the range of interest and for $N \gg 1$

$$R_2(y) \simeq R_1(y) \frac{f^{N+1}(\xi_1) + f^{N+2}\left(\frac{\xi_1 + \xi_2}{2}\right) \frac{\pi}{N(2N+1)}}{f^{N+1}(\xi_1)} \simeq R_1(y) \quad (40)$$

We can therefore see that for relatively long arrays the error in a Woodward synthesis is not substantially different from the Fourier, except at the end of the visible range. If the array is not subject to scanning, this may not be objectionable at all.

For continuous apertures the approximation characteristics of the Woodward method are of similar nature, and it can be shown that the approximating sum converges uniformly to the desired (continuous) pattern. (For a discussion on the continuous aperture approximation see Appendix II.)

IV. EXAMPLES

In order to provide a comparative example for the above methods of approximation a simple pattern was chosen which is easy to obtain by means of any of the above methods, namely a sector beam of $\pm 30^\circ$ beamwidth. Since the beam contains discontinuities, it is a good example of the Gibbs phenomenon as well. If we compare Figure 1 with Figure 2, we can see that the interpolation method yields excellent results except for the overshoot area which seems to suffer even more than the original Fourier approximation. This tends to confirm the original contention that in some cases errors may be worse than those of the Fourier method and in some cases better. Thus it seems that in the continuous range the interpolation method results in a smoother pattern and in the discontinuous it is more oscillatory. Turning now to the Woodward approximation (see Figure 3) it is seen that the error is substantially larger than in the former cases at the end of the visible range while it is quite similar at the origin, in fact as the remainder expression indicated the error at the origin is smaller than either the error of the Fourier method or of the interpolation techniques.

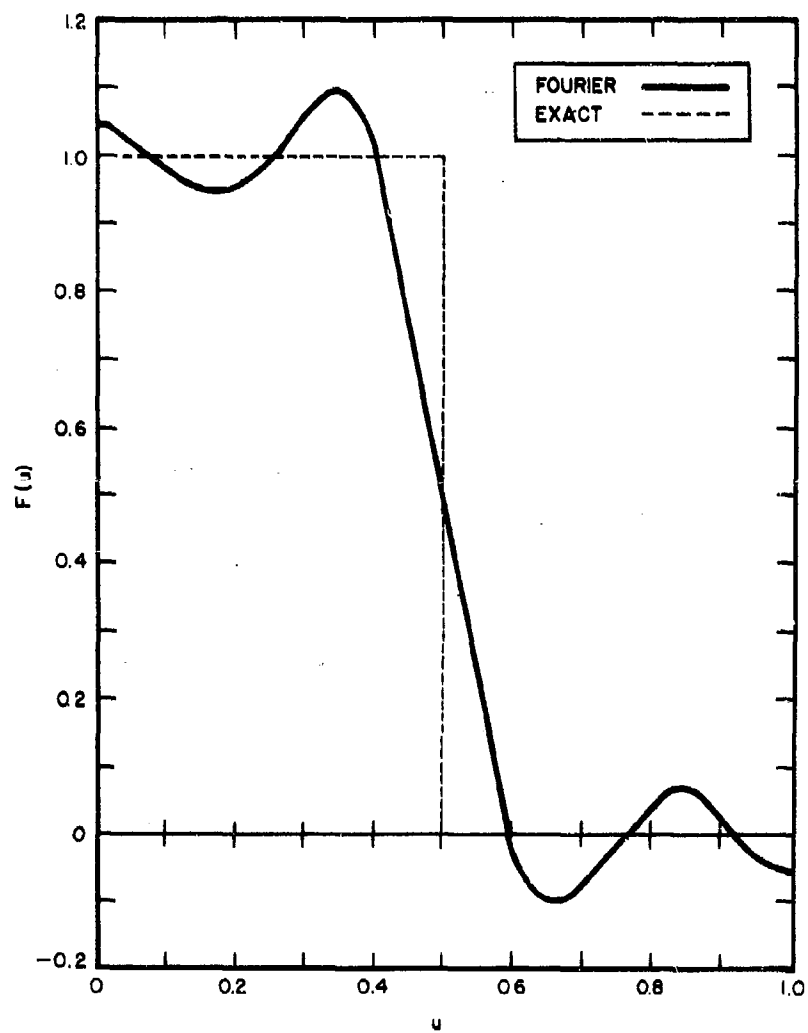


Figure 1. Fourier approximation of a sector beam using 11 elements spaced $\lambda/2$ apart.

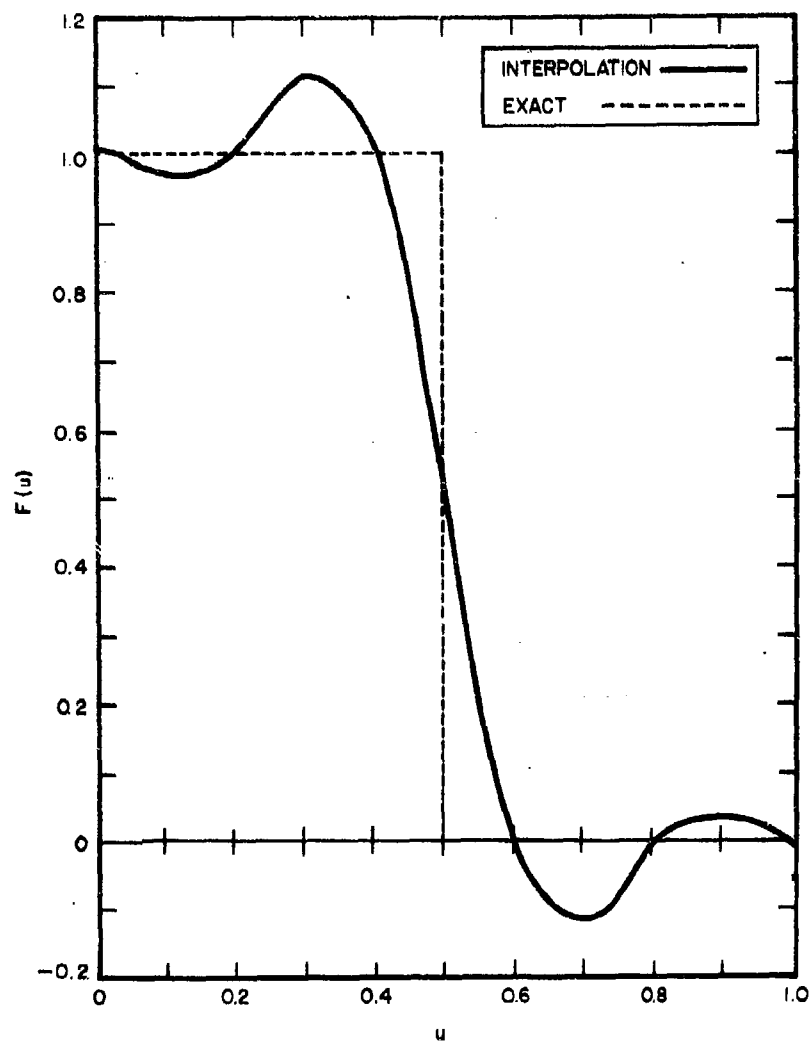


Figure 2. Polynomial interpolation of a sector beam using 11 elements spaced $\lambda/2$ apart.

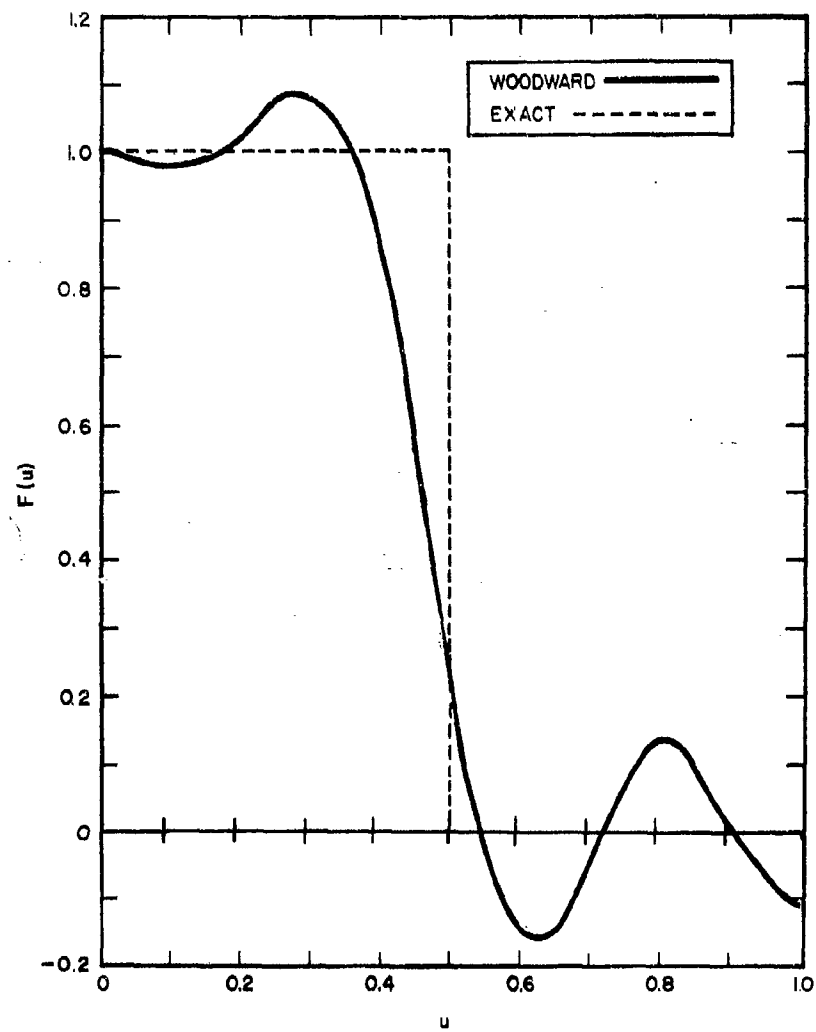


Figure 3. Woodward approximation of a sector beam using 11 elements spaced $\lambda/2$ apart

V. CONCLUSIONS

A method has been presented by means of which an approximation to any desired pattern can be synthesized having error characteristics very similar to those of a Fourier approximation. The method is simpler computationally than the Fourier and consequently is subject to fewer computational errors as well as fewer expenses.

The convergence and error characteristics of the Fourier, the Woodward, and the interpolation methods have been presented and an example has been computed to demonstrate the error behavior. The results indicate that all three methods are adequate in many instances but in some the Woodward method may not be satisfactory.

APPENDIX I

Computation of the Coefficients of a Fourier Interpolation Sum of 6 Terms Approximating a Sector Beam (see Figure 2).

The first coefficient is given by

$$a_0 = \frac{1}{2} \times \frac{2}{5} \sum_{0}^5 y_a = 0.2 \left(\frac{1}{2} + 1 + 1 \right) = 0.5$$

The coefficients of $\cos n\pi u$ are

$$a_1 = \frac{2}{5} \sum y_a \cos \frac{a\pi}{5} = \frac{2}{5} (0.5 + \cos 0.2\pi + \cos 0.4\pi) = 0.646$$

$$a_2 = \frac{2}{5} \sum y_a \cos \frac{2a\pi}{5} = \frac{2}{5} (0.5 + \cos 0.4\pi + \cos 0.8\pi) = 0$$

$$a_3 = \frac{2}{5} \sum y_a \cos \frac{3a\pi}{5} = \frac{2}{5} (0.5 + \cos 0.6\pi + \cos 1.2\pi) = -0.247$$

$$a_4 = \frac{2}{5} \sum y_a \cos \frac{4a\pi}{5} = \frac{2}{5} (0.5 + \cos 0.8\pi + \cos 1.6\pi) = 0$$

$$a_5 = \frac{1}{2} \times \frac{2}{5} \sum y_a \cos \frac{5a\pi}{5} = \frac{1}{2} \times \frac{2}{5} (0.5 + \cos \pi + \cos 2\pi) = 0.100$$

Note that a_0 and a_5 are divided by 2. The resultant expression for the polynomial is given by

$$F(u) = 0.5 + 0.646 \cos 2\pi u - 0.247 \cos 3\pi u + 0.100 \cos 5\pi u.$$

APPENDIX II

Approximate Characteristics of Woodward Synthesis for Continuous Apertures.

The Woodward synthesis is closely related to the sampling theorems in networks, and certain recent mathematical results⁶ obtained in that field can be applied when properly interpreted. For an infinite aperture the pattern can be obtained exactly as long as the transform pair exists,

$$G(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{jxu} dx ; \quad F(x) = \int_{-\infty}^{\infty} G(u) e^{-jxu} du \quad (\text{II-1})$$

where $u = \pi \sin \theta$ and x is measured in half wavelength units ($\frac{d}{\lambda/2}$ where d has the dimension of length), $F(x)$ represents the excitation while $G(u)$ represents the far field pattern. The case of practical interest, however, involves apertures of finite size. One can obtain an upper bound on the error of the Woodward approximation for such cases, and one can also show that this error goes uniformly to zero as the aperture increases indefinitely. The error depends on the behavior of the part of the excitation distribution that is omitted due to the finiteness of the available aperture. We will assume that this excitation attenuates at a certain rate (if it did not attenuate the transform pair of equation (II-1) would not exist) given by

$$\left. \begin{aligned} |F(x)| &< M_1 |x|^{-k_1} \\ |F'(x)| &< M_2 |x|^{-k_2} \end{aligned} \right\} \quad x > W \quad (\text{II-2})$$

where $F'(x)$ is the derivative of $F(x)$, $W\lambda$ is the aperture width, and $M_1 > 0$, $M_2 > 0$, $k_1 > 2$, $k_2 > 2$. Another constant representing the characteristics of the pattern both in the visible and invisible ranges is given by

$$M_3 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u G(u)| du \quad (\text{II-3})$$

The behavior of $G(u)$ outside the visible range may be arbitrarily specified, and since it represents the reactive energy, it may be advantageous to require that it be small. One must remember though, that its specification will affect the conditions given by equation (II-2) which affect the error expressions. The error bound is given by

$$\begin{aligned}
\left| G(u) - \sum_{n=N-1}^{N-1} f\left(\frac{\pi n}{W}\right) \frac{\sin(Wu - \pi n)}{Wu - \pi n} \right| &< 6M_1 W^{1-k_1} + M_2 \frac{k_2+4}{k_2-1} W^{2-k_2} \\
&+ W^2 \left[M_3 + \frac{5M_2}{8(k_2-1)} W^{-k_2} \right] \left[\frac{2\pi}{2N\pi-1} + \frac{1}{N\pi-1} \ln(2N\pi-1) \right] \quad (II-4)
\end{aligned}$$

From equation (II-4) one can see that the third term on the right hand side goes uniformly to zero when N goes to infinity, and W remains fixed. Letting $N > W$ implies that beams are pointed in imaginary directions, that is, the presence of some reactive energy. However, this does not imply super-gain; it just requires that the reactive components of the excitation function be properly adjusted. Under such conditions the error bound is given by the first two terms alone, namely,

$$\text{Error} \leq 6M_1 W^{1-k_1} + M_2 \frac{k_2+4}{k_2-1} W^{2-k_2}$$

which goes to zero as W or the aperture width increases indefinitely.

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